Indian Statistical Institute, Bangalore Centre
B.Math (Hons) II Year, Second Semester
Semestral Examination
Optimization

Time: 3 Hours May 04, 2012 Instructor: Pl.Muthuramalingam
Maximum mark you can get is:50

You can get a maximum marks of 26 in part A

Part A

1. Let $\mathbf{x}$ be as in the primal part and $\mathbf{y}$ be as in the dual part of the conversion table given in the last page. Verify that $\sum_{i,j} y_i a_{ij} x_j \geq \sum_i y_i b_i$. [3]

2. Let $A : R^n_{col} \to R^{m_o}_{col}$, be any linear onto map where $m_0 < n$. Let $\mathbf{a}^1, \mathbf{a}^2, \cdots \mathbf{a}^n$ be the columns of A so that $A = [\mathbf{a}^1, \mathbf{a}^2, \cdots, \mathbf{a}^n]$. Let $B = [\mathbf{a}^1, \mathbf{a}^2, \cdots, \mathbf{a}^{m_0}]$ be an invertible matrix. Fix j with $m_0+1 \leq j \leq n$. Let $\mathbf{a}^j = B\mathbf{r}^j$ for a unique $\mathbf{r}^j$ in $R^{m_0}_{col}$. Let $\mathbf{r}^j = (r^j_1, r^j_2, \cdots, r^j_{m_0})^t$. Assume that $r^j_i \leq 0$ for each i. Let $\mathbf{c} \varepsilon R^n_{col}, \mathbf{c}^t = (c_1, c_2, \cdots, c_n)$. Further, let $c_j < \sum_{q=1}^{m_0} c_q r^j_q$. Let $\mathbf{x}$ be any B basic feasible solution, for the equation $A\ \mathbf{w} = \mathbf{b}$, with $\mathbf{b}$ a given fixed vector. Then show that $\inf\{\mathrm{c}^t \mathbf{v} : A\ \mathbf{v} = \mathbf{b}, \mathbf{v} \geq 0\} = -\infty$.

   [Hint: Choosing $\mathbf{k}$ suitably and considering the family

   $\{\mathbf{x} + \theta\mathbf{k} : \theta$ real $\}$ may help. $\mathbf{k}$ may have entries $-1, 0$, and entries of $\mathbf{r}^j$]. [4]

3. a) <u>*Notation :*</u> For any matrix $A : R^n_{col} \to R^{m_0}_{col}$ define $v_1(A), v_2(A)$

   by $v_1(A) = \max\ \{\min_i \sum a_{ij} x_j : x_j \geq 0, x_1 + x_2 + \cdots + x_n = 1\}$

   and $v_2(A) = \min\ \{\max_j \sum_i y_i a_{ij} : y_i \geq 0, y_1 + y_2 + \cdots + y_{m_0} = 1\}$.

   If $m_0 = n$ and $A^t = -A$ show that $v_1(A) = 0 = v_2(A)$. [3]

   b) For $A : R^n_{col} \to R^{m_0}_{col}$, define $B : R^n_{col} \to R^{m_0}_{col}$ by $b_{ij} = a_{ij} + k$ where k is a fixed real constant. Find a relation between $v_1(B)$ and $v_1(A)$ and prove your claim. [1]

4. Let $g, f : R^2 \longrightarrow R$ be C^2 functions. Let $S = \{\mathbf{x} \varepsilon R^2 : g(\mathbf{x}) = c\}$ where c is a given constant. Define $L : R^2 \times R \longrightarrow R$ by $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$. Assume that $\mathbf{x} \varepsilon S$ is a local minima for f/S. Further assume that rank lin$\cdot$span $\{\bigtriangledown g(\mathbf{x})\} = 1$ for all $\mathbf{y}$ with $\| \mathbf{y} - \mathbf{x} \| < \varepsilon_0$ for some $\varepsilon_0 > 0$. Then we know there exists λ such that $[\bigtriangledown f + \lambda \bigtriangledown g](\mathbf{x}) = 0$. Show that for the same λ, the following second derivative condition is

satisfied. Let $\sum(\mathbf{x}) = \{\mathbf{z}\varepsilon R^2_{col} : g'(\mathbf{x})\mathbf{z} = 0\}$. Then $\mathbf{z}^t L''(\mathbf{x}^*, \lambda)\mathbf{z} \geq 0$ for all $\mathbf{z}$ in $\sum(\mathbf{x}^*)$. Here $L''(\mathbf{x}^*, \lambda) = ((\frac{\partial^2 L}{\partial x_i\, \partial x_j} : (\mathbf{x}^*, \lambda)))$. [7]

5. Let $g_1, g_2, \cdots g_r : R^n \longrightarrow R$ for $r < n$, be C^1 functions. Let $c_1, c_2, \cdots, c_r$ be constants, $(\mathbf{a}, \mathbf{b})\varepsilon R^{n-r} \times R^r$.Assume that $g_i(\mathbf{a}, \mathbf{b}) = c_i$ for each i. Let $\mathbf{a}\varepsilon U$, U open set in R^{n-r}. Assume that $h_j : U \to R$ for $j = n-r+1, n-r+2, \cdots, n$ satisfy $(h_{n-r+1}(\mathbf{a}), \cdots h_n(\mathbf{a})) = \mathbf{b}$ and $g_i(\mathbf{z}, h_{n-r+1}(\mathbf{z}), \cdots h_n(\mathbf{z})) = c_i$. for $\mathbf{z}$ in U. Then show that, with $\partial_i = \frac{\partial}{\partial x_i}$,

$$\begin{pmatrix} \partial_1 g_1 & \partial_2 g_2 & \cdot\cdot & \partial_n g_1 \\ \partial_1 g_2 & \partial_2 g_2 & \cdot\cdot & \partial_n g_2 \\ \\ \partial_1 g_r & \partial_2 g_r & \cdot\cdot & \partial_n g_r \end{pmatrix} (\mathbf{a}, \mathbf{b}) \begin{pmatrix} 1\cdot & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 1\cdot \\ \partial_1 h_{n-r+1} & \partial_2 h_{n-r+1} & \cdots \partial_{n-r} h_{n-r+1} \\ \partial_1 h_{n-r+2} & \partial_2 h_{n-r+2} & \cdots \partial_{n-r} h_{n-r+2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \partial_1 h_n & \partial_2 h_n & \cdots \partial_{n-r} h_n \end{pmatrix}$$

[2]

6. Let $f, g : R \to R$ be C^1 functions, $D = \{x : g(x) \geq 0\}$, and $g(x_0) = 0$. If x_0 is local maxima for $f\ |_D$, $g'(x_0) < 0$ and $f'(x_0) \neq 0$, then show that $f'(x_o) > 0$. [2]

7. Let $f : D \longrightarrow R$ be a concave function on a convex set D of R^n. If $\mathbf{x}^0$ is an intrior point of D and is a local maxima for f, then show that $f(\mathbf{x}^0) = \max\limits_D f$. [3]

8. Let $g_1, g_2, \cdots g_r, f : R^n \to R$ be concave C^1 functions. Let $D = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0$ for each $j\}$ be nonempty. Let $M_j \geq 0$. Let $\mathbf{x}^*\varepsilon D$.

   a) Show that D is a convex set. [1]

   b) Show that $f + \sum M_j g_j$ is a concave function. [1]

   c) Let $\sum M_j g_j[\mathbf{x}^*] = 0$ and $[\bigtriangledown f + \sum M_j \bigtriangledown g_j](\mathbf{x}^*) = 0$. Then show that $f(\mathbf{x}^*) = \max\limits_D$ f. [4]

Part B

9. Determine the maximum value of $18x_1 + 4x_2 + 6x_3$ under the constraints

   $3x_1 + x_2 \leq -3$

   $2x_1 + x_3 \leq -5$

$x_1 \leq 0, x_2 \leq 0, x_3 \leq 0$

by looking at the dual problem or directly. [4]

10. A factory or firm produces two outputs y and z using a single input x. The set of attainable output levels $H(x)$ form an input use of x is given by $H(x) = \{(y, z) : y^2 + z^2 \leq x\}$. The firm has available to it a maximum of one unit of input x. Let p_1, p_2 denote the price of y, z respectively. Determine the firms optimal output mix, using Kuha-Tucker theorem. Also find the maximum selling price.

    [Hint: Let $f, g_1, g_2, g_3, g_4, g_5 : R^3 \to R$ be given by

$$
\begin{aligned}
f(x, y, z) &= p_1 y + p_2 z, \\
g_1(x, y, z) &= x - (y^2 + z^2), \\
g_2(x, y, z) &= x, \\
g_3(x, y, z) &= 1 - x, \\
g_4(x, y, z) &= y, \\
g_5(x, y, z) &= z,
\end{aligned}
$$

    $S = \{(x, y, z) : g_i(x, y, z) \geq 0 \text{ for each } i\}$. If (x^*, y^*, z^*) is local maxima for f on S, you can assume for (obvious?) reasons $g_2(x^*, y^*, z^*) > 0, g_4(x^*, y^*, z^*) > 0$ and $g_5(x^*, y^*, z^*) > 0$. [10]

11. a) Let $A = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}$. Calculate $v_1(A)$. [3]

    b) Let $B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. Calculate $v_2(B)$. [1]

    [Hint you may use a portion of question 3 of part A]

12. Let $g, f : R \longrightarrow R$ be given by

    $f(x) = x^n \quad n \varepsilon \{2, 3, 4, \cdots\}$,
    $g(x) = x$.

    Let $D = \{g \geq 0\}$ and $x^* = 0, \lambda = 0$.

    a) Show that x^* is not a local maxima for f on D

    b) $x^* \varepsilon D, g(x^*) = 0$, rank lin span $\{\bigtriangledown g(x^*)\} = 1, \lambda \geq 0, \lambda g(x^*) = 0$.

    c) $(\bigtriangledown f + \lambda \bigtriangledown g)(x^*) = 0$. [3]

13. Let $f : (0, \infty) \times (0, \infty) \to R$ be given by $f(x, y) = x^a y^b, a > 0, b > 0$. If $a + b \leq 1$, then f is concave function. [3]

Full Conversion Table

	Primal	Dual
	$A, \mathbf{x}, \mathbf{b}, \mathbf{c}$	$A^t, \mathbf{y}^t, \mathbf{c}^t, \mathbf{b}^t$
$i\varepsilon I_1,$	$\sum_j a_{ij}x_j = b_i$	$y_i real, y_i \gtrless 0$
$i\varepsilon I_2,$	$\sum_j a_{ij}x_j \geq b_i$	$y_i \geq 0$
$i\varepsilon I_3,$	$\sum_j a_{ij}x_j \leq b_i$	$y_i \leq 0$
$j\varepsilon J_1,$	$x_j real, x_j \gtrless 0$	$\sum_i y_i a_{ij} = c_j$
$j\varepsilon J_2,$	$x_j \geq 0$	$\sum_i y_i a_{ij} \leq c_j$
$j\varepsilon J_3,$	$x_j \leq 0$	$\sum y_i a_{ij} \geq c_j$
	$min \sum_j c_j x_j$	$max \sum_i y_i b_i$